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On the representations of $GL_q(n)$ using the Heisenberg–Weyl relations

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Abstract. Using the representations of the Heisenberg–Weyl relations we develop a systematic scheme for constructing finite and infinite dimensional representations of the elements of the quantum groups $GL_q(n)$, where the deformation parameter q is a primitive root of unity. Explicit results for the examples $GL_q(2)$, $GL_q(3)$ and $GL_q(4)$ are discussed.

1. Introduction

Quantum groups (Drinfeld 1985, Jimbo 1986) appear as an underlying mathematical structure in several contexts; namely, quantum inverse scattering methods, the rational conformal field theory and the theory of braids (Faddeev *et al* 1987, Takhtajan 1989, Alvarez-Gaumé *et al* 1989 and the references therein). These constructs may be viewed as matrix groups with the non-commutative elements obeying sets of bilinear product relations, as well as deformations of the Lie algebras. The sufficient condition for the associativity of the algebras turns out to be the Yang-Baxter relation (Yang 1967, Baxter 1982), the analogue for the Jacobi identity for the quantum groups.

In the viewpoint considered by Manin (1988) and other authors (Vokos *et al* 1989, Corrigan *et al* 1990) a quantum group is identified with the endomorphisms acting on spaces whose elements are non-commuting coordinates. In the matrix representations of the endomorphisms, the commutation relations for the space coordinates generate the commutation relations the matrix elements have to satisfy. The minimal set of relations imposed by Manin's construction turn out to be same as the set of bilinear relations specified by an R-matrix satisfying the Yang–Baxter equation. Viewed alternately, the structure of the R-matrix may be understood by considering the commutation relations imposed by Manin's construction.

Following Corrigan *et al* (1990) we enlist (2.4) the commutation relations for $GL_q(n)$. A subset of bilinear product relations for $GL_q(n)$ have the form

$$m_A m_B = q^{n_{A,B}} m_B m_A \tag{1.1}$$

where the pairs (m_A, m_B) refer to a subset of pairs of the elements of $GL_q(n)$ and q is the deformation parameter. For unimodular values of q (for us, throughout, q is a primitive Nth root of unity), the relations (1.1) are of the Heisenberg–Weyl type. The remaining bilinear relations for $GL_q(n)$ may be reformulated (Floratos 1989, Weyers 1990) in the Heisenberg–Weyl form (1.1) provided some invertibility conditions are

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satisfied. Exploiting the representation of the Heisenberg–Weyl group in the discrete space of N-vertices of a canonical polygon on a plane, Floratos (1989) constructed a matrix representation of the elements of $GL_q(2)$ and Weyers (1990) noticed the possibility of using this technique to obtain the representations of the elements of $GL_q(n)$ for an arbitrary n.

In this article we discuss a general algorithm for constructing the finite and infinite dimensional representations of the elements of $GL_q(n)$; the examples of $GL_q(2)$ and $GL_q(3)$ considered earlier (Floratos 1989, Weyers 1990) may be treated as special cases of our general prescription. To this end we make use of the elements of representation theory of generalized Clifford algebras (Ramakrishnan 1972, Jagannathan and Ranganathan 1974, 1975, Ramakrishnan and Jagannathan 1976, Jagannathan 1985 and the references therein); for these algebras, relations of the type (1.1) form part of the generating relations. The essential idea is as follows. For $GL_q(n)$ the $n^2 \times n^2$ antisymmetric integer matrix P of the exponents $[n_{A,B}]$ (where $n_{A,B} = -n_{B,A}$) may be related to its unique skew-normal form $\rho = [\nu_{A,B}]$ through a transformation

$$P = U^{\mathrm{T}} \rho U \tag{1.2}$$

where U is a unimodular ($|\det U| = 1$) integer matrix (see Newman (1972) for the construction of U and ρ for a given P). We may construct (Ramakrishnan and Jagannathan 1976) the elements m_A as a 'product transformation' of another set (μ_A),

$$m_A = \prod_B \mu_B^{u_{BA}} \tag{1.3}$$

where $[u_{AB}] = U$. A direct computation verifies that the elements m_A constructed by the above procedure satisfy the Heisenberg–Weyl relations (1.1) provided the set (μ_A) exhibit the bilinear product relations

$$\mu_A \mu_B = q^{\nu_{A,B}} \mu_B \mu_A. \tag{1.4}$$

The matrix $\rho = [\nu_{A,B}]$ is unique (see Newman 1972) and has the structure

$$\rho = \begin{pmatrix} 0 & \nu_{1,2} \\ -\nu_{1,2} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \nu_{3,4} \\ -\nu_{3,4} & 0 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 & \nu_{2s-1,2s} \\ -\nu_{2s-1,2s} & 0 \end{pmatrix} \oplus O_{n\times n}. \tag{1.5}$$

The rank of the matrix ρ is $2s$ (where $s = \binom{n}{2}$) and $\nu_{2j-1,2j}$ divides $\nu_{2j+1,2j+2}$ for each $j = 1, 2, \ldots, s-1$. This leads to a realization of m_A as tensor product of operators acting on s subspaces. The representation of the set (m_A) achieved here is, in general, not unique as two alternate unimodular matrices, say U and U', would yield two distinct sets of m_A exhibiting the same commutation relations (1.1) provided $U' = VU$ and $V^{\mathrm{T}} \rho V = \rho$. We notice that if all μ_A are represented by non-singular matrices then the representation of the set (m_A) described above, however, is unique under equivalence up to constant multiplication factors since the set (m_A) generates a projective representation of a finite Abelian group (Backhouse and Bradley 1972, Morris 1973 and the references therein).

The plan of the article is as follows. Mainly to specify our notations, we review certain aspects of the quantum group $GL_q(n)$ in section 2. Sections 3, 4 and 5 contain our discussions for the examples of $GL_q(2)$, $GL_q(3)$ and $GL_q(4)$, respectively. We conclude in section 6.

2. Manin's construction for $GL_q(n)$

Manin (1988) introduced an n-dimensional vector space $X (\in R_q(n.0)) = (X_i)$, $1 \leq i \leq n$, the coordinates of which satisfy the bilinear product relations

$$X_iX_j = q^{-1}X_jX_i \qquad \text{for } i<j. \tag{2.1}$$

A dual quantum vector space $\xi(\in R_q(0,n)) = (\xi_i)$, $1 \leqslant i \leqslant n$ with the coordinates satisfying the relations

$$\begin{aligned} &\xi_i^2 = 0 \\ &\xi_i\xi_j + q\xi_j\xi_i = 0 \qquad \text{for } i<j \end{aligned} \tag{2.2}$$

is also introduced. The quantum group may then be viewed as a linear transformation matrix $M(\in \mathrm{GL}_q(n)) = (m_{ij})$, $1 \leqslant i, j \leqslant n$ of these vector spaces,

$$X' = MX \qquad \xi' = M\xi \tag{2.3}$$

which preserves the bilinear product relations (2.1) and (2.2). The requirement $X' \in R_q(n,0)$ and $\xi' \in R_q(0,n)$ induces the the following bilinear relations (Corrigan *et al* 1990) for the elements m_{ij}:

$$m_{ij}m_{ik} = q^{-1}m_{ik}m_{ij} \qquad j<k \tag{2.4a}$$

$$m_{ik}m_{jk} = q^{-1}m_{jk}m_{ik} \qquad i<j \tag{2.4b}$$

$$m_{ij}m_{kl} = m_{kl}m_{ij} \qquad i<k, j>l \tag{2.4c}$$

$$m_{ij}m_{kl} - m_{kl}m_{ij} = (q^{-1}-q)m_{il}m_{kj} \qquad i<k, j<l. \tag{2.4d}$$

The quantum determinant $\mathrm{D}_q(M)$ of the matrix M is defined recursively:

$$\begin{aligned} &\mathrm{D}_q(m_{ij}) = m_{ij} \\ &\mathrm{D}_q(M) = \sum_{i=1}^{n} (-1)^{n-1}q^{n-1}m_{1i}\mathrm{D}_q(M_{1i}). \end{aligned} \tag{2.5}$$

The quantum determinant $\mathrm{D}_q(M)$ has the property that it commutes with all the matrix elements of M,

$$m_{ij}\mathrm{D}_q(M) = \mathrm{D}_q(M)m_{ij} \tag{2.6}$$

and therefore $\mathrm{D}_q(M)$ is a central element.

The bilinear product relationships (2.4) may also be understood as the relations dictated by an R-matrix condition for the quantum group $\mathrm{GL}_q(n)$:

$$R_{ij,kl}M_{km}M_{ln} = M_{jl}M_{ik}R_{kl,mn} \tag{2.7}$$

where the R-matrix is given by (Corrigan *et al* 1990),

$$R_{ij,kl} = \delta_{ik}\delta_{jl}(1+(q^{-1}-1)\delta_{ij}) + (q^{-1}-q)\theta(i-k)\delta_{il}\delta_{jk} \tag{2.8}$$

with the step function $\theta(i-k)$ defined as

$$\theta(i-k) = \begin{cases} 1 & i>k \\ 0 & i \leqslant k. \end{cases} \tag{2.9}$$

The R-matrix is a linear transformation acting on a direct product vector space $V^{\otimes 2}$. In an alternate viewpoint the structure of the R-matrix (2.8) may be inferred from the defining relation (2.7) and the bilinear commutation relations (2.4) obtained through Manin's construction. The R-matrix thus obtained satisfies the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{2.10}$$

as a sufficient condition for the associativity. The notation R_{ij} denotes an operator acting on a triple tensor product of vector spaces $V_i \otimes V_j \otimes V_k$ such that its action on $V_i \otimes V_j$ is described by the R-matrix and its action on V_k reduces to the identity.

Floratos (1989) and Weyers (1990) noted that the bilinear product relations (2.4a–c) are of the Heisenberg–Weyl type. These authors proved that, by making a suitable choice of the variables, all the product relations for $\mathrm{GL}_q(n)$ may be recast as the Heisenberg–Weyl relations, provided that the inverses of certain elements exist. We enlist our choice of the 'Heisenberg–Weyl variables' for the cases $\mathrm{GL}_q(2)$, $\mathrm{GL}_q(3)$ and $\mathrm{GL}_q(4)$ in (3.1), (4.1) and (5.1), respectively.

3. Representation for $\mathrm{GL}_q(2)$

For the case of $\mathrm{GL}_q(2)$ we choose the Heisenberg–Weyl variables

$$m_{12}, m_{21}, m_{22} \text{ and } \mathrm{D}_q(M). \tag{3.1}$$

The remaining element m_{11} of the matrix M may be solved as

$$m_{11} = (\mathrm{D}_q(M) + q^{-1} m_{12} m_{21}) m_{22}^{-1} \tag{3.2}$$

provided m_{22}^{-1} exists. For the two arbitrary elements m_A and m_B in the list (3.1), the product relation is of the form (1.1), where the exponents $n_{A,B}$ are listed in table 1.

Table 1. The matrix $P = [n_{A,B}]$ for $\mathrm{GL}_q(2)$.

		m_B			
		m_{12}	m_{21}	m_{22}	$\mathrm{D}_q(M)$
m_A	m_{12}	0	0	−1	0
	m_{21}	0	0	−1	0
	m_{22}	1	1	0	0
	$\mathrm{D}_q(M)$	0	0	0	0

(3.3)

In the notation introduced earlier, the entries of table 1 form the P-matrix for the $\mathrm{GL}_q(2)$ case. The rank of the P-matrix is 2, as may be directly seen from its structure. The corresponding canonical form ρ and the unimodular integer matrix U are given by

$$\rho = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.4}$$

and

$$U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.5}$$

respectively. The structure of the ρ suggests the following choice of the commutation rules for μ_A $(A = 1, \ldots, 4)$:

$$\mu_1 \mu_2 = q^{-1} \mu_2 \mu_1 \tag{3.6}$$

and (μ_3, μ_4) are central elements. Following (1.3) and (3.5) we may now express the Heisenberg–Weyl set of variables listed in (3.1)

$$m_{12}=\mu_1 \qquad m_{21}=\mu_1\mu_3 \qquad m_{22}=\mu_2 \qquad \mathrm{D}_q(M)=\mu_4. \tag{3.7}$$

When q is a primitive Nth root of unity we make the following choice without any loss of generality

$$\mu_1=h_N(K) \qquad \mu_2=\chi g_N(q) \tag{3.8}$$

where

$$h_N(K)=\begin{bmatrix} 0 & k_1 & 0 & \dots & 0 \\ 0 & 0 & k_2 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & \dots & k_{N-1} \\ k_N & 0 & \dots & \dots & 0 \end{bmatrix} \tag{3.9}$$

$$g_N(q)=\begin{bmatrix} 1 & & & & 0 \\ & q^{-1} & & & \\ & & q^{-2} & & \\ & 0 & & \ddots & \\ & & & & q^{-(N-1)} \end{bmatrix} \tag{3.10}$$

$K=(k_1,\dots,k_N)$. The parameters k_i ($i=1,2,\dots,N$) and $\chi(\neq 0)$ are arbitrary complex numbers. Our analysis essentially reproduces the results of Floratos (1989). The equivalence may be established by taking the finite Fourier transformation of the corresponding results. The elements μ_3 and μ_4 are central in the Heisenberg–Weyl group and therefore must be constants, say C_1 and C_2 respectively. This finally leads to the representation

$$\begin{aligned} m_{12}&=h_N(K) & m_{21}&=C_1h_N(K) \\ m_{22}&=\chi g_N(q) & m_{11}&=\chi^{-1}(C_2+q^{-1}C_1h_N(K)^2)g_N(q)^{-1}. \end{aligned} \tag{3.11}$$

We obtained the representation for m_{11} in (3.11) by using (3.2). Notice that while Floratos (1989) made a choice of m_{11} to be invertible, we considered m_{22} to exhibit invertibility.

In the case of projective representations of finite Abelian groups (Weyl 1950) or generalized Clifford algebras, μ_1 and μ_2 are taken to be invertible, satisfying

$$\mu_1^N=\mu_2^N=1. \tag{3.12}$$

Then, necessarily we have

$$k_1k_2\dots k_N=1 \tag{3.13}$$

and all such representations are equivalent to one in which

$$k_1=k_2=\dots=k_N=1 \qquad \chi=1. \tag{3.14}$$

In the present context of bilinear relations for the elements of the matrix representation of the quantum groups the condition (3.12) is absent and as a consequence arbitrary constants (k_i) and χ appear in (3.8). If we consider μ_1 and μ_2 to be invertible without imposing the normalization condition (3.12), we may choose the parameters as $k_1 = k_2 = \ldots = k_n = k(\neq 0)$; alternate representations of (μ_1, μ_2) are equivalent up to constant multiplicative factors.

4. Representation for $\mathrm{GL}_q(3)$

The complete set of the Heisenberg–Weyl variables for the quantum group $\mathrm{GL}_q(3)$ is

$$m_{12}, m_{13}, m_{21}, m_{22}, m_{31}, \mathrm{D}_q(M_{33}), \mathrm{D}_q(M_{31}), \mathrm{D}_q(M_{13}) \text{ and } \mathrm{D}_q(M). \tag{4.1}$$

The rest of the elements of the matrix M may be solved as follows:

$$\begin{aligned}
m_{11} &= (\mathrm{D}_q(M_{33}) + q^{-1} m_{12} m_{21}) m_{22}^{-1} \\
m_{23} &= m_{12}^{-1} (\mathrm{D}_q(M_{31}) + q^{-1} m_{13} m_{22}) \\
m_{32} &= m_{21}^{-1} (\mathrm{D}_q(M_{13}) + q^{-1} m_{22} m_{31}) \\
m_{33} &= \mathrm{D}_q(M_{33})^{-1} (\mathrm{D}_q(M) + q^{-1} \mathrm{D}_q(M_{23}) m_{23} - q^{-2} \mathrm{D}_q(M_{13}) m_{13})
\end{aligned} \tag{4.2}$$

provided the inverses of m_{22}, m_{12}, m_{21} and $\mathrm{D}_q(M_{33})$ exist. Two arbitrary elements m_A and m_B in (4.1) satisfy (1.1) and the exponent matrix $P = [n_{A,B}]$ is listed in table 2.

Table 2. The matrix $P = [n_{A,B}]$ for $\mathrm{GL}_q(3)$.

		m_B								
		m_{12}	m_{13}	m_{21}	m_{22}	m_{31}	$\mathrm{D}_q(M_{33})$	$\mathrm{D}_q(M_{31})$	$\mathrm{D}_q(M_{13})$	$\mathrm{D}_q(M)$
m_A	m_{12}	0	−1	0	−1	0	0	0	−1	0
	m_{13}	1	0	0	0	0	0	0	0	0
	m_{21}	0	0	0	−1	−1	0	−1	0	0
	m_{22}	1	0	1	0	0	0	0	0	0
	m_{31}	0	0	1	0	0	1	0	0	0
	$\mathrm{D}_q(M_{33})$	0	−1	0	0	−1	0	−1	−1	0
	$\mathrm{D}_q(M_{31})$	0	0	1	0	0	1	0	0	0
	$\mathrm{D}_q(M_{13})$	1	0	0	0	0	1	0	0	0
	$\mathrm{D}_q(M)$	0	0	0	0	0	0	0	0	0

(4.3)

A direct examination shows that the rank of the P-matrix is 6. The element $\mathrm{D}_q(M)$ is central in nature and the elements of the commuting pairs $(m_{13}, \mathrm{D}_q(M_{13}))$ and $(m_{31}, \mathrm{D}_q(M_{31}))$ have identical commutation relations with other elements. Each such pair decreases the rank of the P-matrix by one. The P-matrix listed in (4.3) leads to the following ρ and U matrices:

$$\rho = \begin{bmatrix}
0 & -1 & & & & & & & \\
1 & 0 & & & & & & & \\
& & 0 & -1 & & & \mathbf{0} & & \\
& & 1 & 0 & & & & & \\
& & & & 0 & 2 & & & \\
& & & & -2 & 0 & & & \\
& & \mathbf{0} & & & & 0 & & \\
& & & & & & & 0 & \\
& & & & & & & & 0
\end{bmatrix} \tag{4.4}$$

$$U=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.5}$$

The structure of the ρ matrix leads to the following set of commutation relations for μ_A $(A=1,\ldots,9)$:

$$\mu_1\mu_2=q^{-1}\mu_2\mu_1 \tag{4.6a}$$

$$\mu_3\mu_4=q^{-1}\mu_4\mu_3 \tag{4.6b}$$

$$\mu_5\mu_6=q^{2}\mu_6\mu_5 \tag{4.6c}$$

$$\mu_A\mu_B=\mu_B\mu_A \qquad \text{otherwise.} \tag{4.6d}$$

The commutation relations (4.6) suggest that the quantum group $GL_q(3)$ may be viewed—apart from the central elements—as $n(n-1)/2$ (where $n=3$) mutually commuting pairs of variables exhibiting Heisenberg–Weyl relationships among themselves. As a generalization in the case of $GL_q(n)$, we may also point out that the structure of (2.4) suggests that the number of mutually commuting pairs therein equals $n(n-1)/2$. The property (4.6) will allow us to construct a representation of μ_A $(A=1,\ldots,9)$ by taking a tensor product of the matrices characteristic of the representations of $GL_q(2)$. This feature will be repeated for the larger quantum groups.

Using (1.3) and the structure of the U-matrix (4.5) we directly construct

$$\begin{aligned} &m_{12}=\mu_1 \qquad m_{13}=\mu_2 \\ &m_{21}=\mu_3 \qquad m_{22}=\mu_2\mu_4 \\ &m_{31}=\mu_4\mu_5 \qquad D_q(M_{33})=\mu_1\mu_3^{N-1}\mu_6 \\ &D_q(M_{31})=\mu_4\mu_5\mu_7 \qquad D_q(M_{13})=\mu_2\mu_8 \\ &D_q(M)=\mu_9. \end{aligned} \tag{4.7}$$

The representations of μ_A $(A=1,\ldots,9)$ may be obtained by considering (3.8) in tensor form. Apart from constant non-zero multiplicative factors, we enlist

$$\begin{aligned} &\mu_1=h_N(K^{(1)})\otimes I_N\otimes I_{N'} \\ &\mu_2=g_N(q)\otimes I_N\otimes I_{N'} \\ &\mu_3=I_N\otimes h_N(K^{(2)})\otimes I_{N'} \\ &\mu_4=I_N\otimes g_N(q)\otimes I_{N'} \\ &\mu_5=I_N\otimes I_N\otimes h_{N'}(K^{(3)}) \\ &\mu_6=I_N\otimes I_N\otimes g_{N'}^{N'-2+r}(q^{1+r}) \\ &\mu_7=\mu_8=\mu_9=I_N\otimes I_N\otimes I_{N'} \end{aligned} \tag{4.8}$$

where

$$N' = \begin{cases} N & \text{for odd } N \\ N/2 & \text{for even } N \end{cases}$$

$$\varepsilon = \begin{cases} 0 & \text{for odd } N \\ 1 & \text{for even } N \end{cases}$$

and $K^{(i)}$ for $i=(1,2)$ $(i=3)$ refer to an N (N') component set of integer parameters. Notice that the requirement of existence of the inverses for m_{12} and m_{21} leads to the restriction $k_i^{(1)}=k^{(1)}\neq 0$ (for $i=1,2,\ldots,N$) and $k_i^{(2)}=k^{(2)}\neq 0$ (for $i=1,2,\ldots,N$) whereas the components of $K^{(3)}$ may be chosen either all equal to some non-zero number or having completely arbitrary values including zero. Since in (4.8) we are specifying the representation only up to constant multiplicative factors we may take $k^{(1)}=k^{(2)}=1$ without any loss of generality. Also, while constructing the matrix representation of μ_A $(A=1,\ldots,9)$ we use only the positive powers of $h_N(K)$ and g_N; this exploits our choice of q to be a primitive Nth root of unity. This is essential whenever the existence of inverses of $h_N(K)$ is not required by construction and consequently K may have components with arbitrary entries including zero. The representations for m_{11}, m_{23}, m_{32} and m_{33} may be directly obtained by substituting (4.7) in (4.2).

5. Representation for $GL_q(4)$

Having explained above our method in detail for the examples of $GL_q(2)$ and $GL_q(3)$, here we will just quote our results for $GL_q(4)$. We choose the following set of Heisenberg–Weyl variables for $GL_q(4)$:

$$m_{12}, m_{13}, m_{14}, m_{21}, m_{22}, m_{31}, m_{41}, D_q(M_{44,33}), D_q(M_{44,31}), D_q(M_{44,13}), D_q(M_{44}),$$
$$D_q(M_{41,32}), D_q(M_{14,23}), D_q(M_{41}), D_q(M_{14}) \text{ and } D_q(M). \tag{5.1}$$

The remaining elements of M may be solved as follows:

$$\begin{aligned}
m_{11} &= (D_q(M_{44,33})+q^{-1}m_{12}m_{21})m_{22}^{-1} \\
m_{23} &= m_{12}^{-1}(D_q(M_{44,31})+q^{-1}m_{13}m_{22}) \\
m_{32} &= m_{21}^{-1}(D_q(M_{44,13})+q^{-1}m_{22}m_{31}) \\
m_{33} &= D_q(M_{44,33})^{-1}(D_q(M_{44})+q^{-1}D_q(M_{44,23})m_{23}-q^{-2}D_q(M_{44,13})m_{13}) \\
m_{24} &= m_{13}^{-1}(D_q(M_{41,32})+q^{-1}m_{14}m_{23}) \\
m_{42} &= m_{31}^{-1}(D_q(M_{14,23})+q^{-1}m_{32}m_{41}) \\
m_{34} &= D_q(M_{44,31})^{-1}(D_q(M_{41})+q^{-1}D_q(M_{41,24})m_{24}-q^{-2}D_q(M_{41,14})m_{14}) \\
m_{43} &= D_q(M_{44,13})^{-1}(D_q(M_{14})+q^{-1}D_q(M_{14,33})m_{33}-q^{-2}D_q(M_{14,23})m_{23}) \\
m_{44} &= D_q(M_{44})^{-1}(D_q(M)+q^{-1}D_q(M_{34})m_{34}-q^{-2}D_q(M_{24})m_{24}+q^{-3}D_q(M_{14})m_{14})
\end{aligned} \tag{5.2}$$

provided the inverses of m_{22}, m_{12}, m_{21}, $D_q(M_{44,33})$, m_{13}, m_{31}, $D_q(M_{44,31})$, $D_q(M_{44,13})$ and $D_q(M_{44})$ exist. Table 3 lists the exponent matrix $P=[n_{A,B}]$ (where $A, B=1,\ldots,16$) for the Heisenberg–Weyl variables listed in (5.1). The corresponding ρ and U matrices

Table 3. The matrix $P=[n_{A,B}]$ for $GL_q(4)$.

		m_B															
		m_{12}	m_{13}	m_{14}	m_{21}	m_{22}	m_{31}	m_{41}	$D_q(M_{44,33})$	$D_q(M_{44,31})$	$D_q(M_{44,13})$	$D_q(M_{44})$	$D_q(M_{41,32})$	$D_q(M_{14,23})$	$D_q(M_{41})$	$D_q(M_{14})$	$D_q(M)$
m_A	m_{12}	0	−1	−1	0	−1	0	0	0	0	−1	0	−1	−1	0	−1	0
	m_{13}	1	0	−1	0	0	0	0	1	0	0	0	0	0	0	−1	0
	m_{14}	1	1	0	0	0	0	0	1	1	0	1	0	0	0	0	0
	m_{21}	0	0	0	0	−1	−1	−1	0	−1	0	0	−1	−1	−1	0	0
	m_{22}	1	0	0	1	0	0	0	0	0	0	0	−1	−1	0	0	0
	m_{31}	0	0	0	1	0	0	−1	1	0	0	0	0	0	−1	0	0
	m_{41}	0	0	0	1	0	1	0	1	0	1	1	0	0	0	0	0
	$D_q(M_{44,33})$	0	−1	−1	0	0	−1	−1	0	−1	−1	0	−2	−2	−1	−1	0
	$D_q(M_{44,31})$	0	0	−1	1	0	0	0	1	0	0	0	−1	−1	0	−1	0
	$D_q(M_{44,13})$	1	0	0	0	0	0	−1	1	0	0	0	−1	−1	−1	0	0
	$D_q(M_{44})$	0	0	−1	0	0	0	−1	0	0	0	0	−1	−1	−1	−1	0
	$D_q(M_{41,32})$	1	0	0	1	1	0	0	2	1	1	1	0	0	0	0	0
	$D_q(M_{14,23})$	1	0	0	1	1	0	0	2	1	1	1	0	0	0	0	0
	$D_q(M_{41})$	0	0	0	1	0	1	0	1	0	1	1	0	0	0	0	0
	$D_q(M_{14})$	1	1	0	0	0	0	0	1	1	0	1	0	0	0	0	0
	$D_q(M)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(5.3)

are respectively:

$$\rho=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\oplus\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\oplus\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\oplus\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix}\oplus\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix}\oplus\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix}\oplus O_{4\times 4} \quad (5.4)$$

$$U=\begin{bmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & -1 & 1 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

The structure of the ρ matrix in (5.4) immediately indicates the following commutation relations for μ_A $(A=1,\ldots,16)$:

$$\begin{aligned}
&\mu_1\mu_2=q^{-1}\mu_2\mu_1\\
&\mu_3\mu_4=q^{-1}\mu_4\mu_3\\
&\mu_5\mu_6=q^{-1}\mu_6\mu_5\\
&\mu_7\mu_8=q^2\mu_8\mu_7\\
&\mu_9\mu_{10}=q^2\mu_{10}\mu_9\\
&\mu_{11}\mu_{12}=q^2\mu_{12}\mu_{11}\\
&\mu_A\mu_B=\mu_B\mu_A \qquad \text{otherwise.}
\end{aligned} \tag{5.6}$$

Equation (5.6) indicates six pairs of mutually commuting variables exhibiting the Heisenberg–Weyl relationship within each pair. This agrees with the discussion following (4.6).

The structure of U in (5.5) and the construction of m_A $(A=1,\ldots,16)$ in (1.3) immediately yields

$$\begin{aligned}
&m_{12}=\mu_1 \qquad m_{13}=\mu_2 \qquad m_{14}=\mu_1^{N-1}\mu_2\mu_3\\
&m_{21}=\mu_3\mu_5 \qquad m_{22}=\mu_2\mu_4 \qquad m_{31}=\mu_6\\
&m_{41}=\mu_5^{N-1}\mu_6\mu_7 \qquad \mathrm{D}_q(M_{44,33})=\mu_1\mu_3^{N-1}\mu_5\mu_7^{N'-1}\mu_{10}\\
&\mathrm{D}_q(M_{44,31})=\mu_4^{N-1}\mu_6^2\mu_8^{N'-1}\mu_{10}\mu_{12}\\
&\mathrm{D}_q(M_{44,13})=\mu_2\mu_4\mu_6^{N-1}\mu_8\\
&\mathrm{D}_q(M_{44})=\mu_4^{N-1}\mu_6\mu_9^{N'-1}\mu_{10}\mu_{11}\\
&\mathrm{D}_q(M_{41,32})=\mu_2\mu_3^{N-1}\mu_4\mu_9\\
&\mathrm{D}_q(M_{14,23})=\mu_2\mu_3^{N-1}\mu_4\mu_9\mu_{13}\\
&\mathrm{D}_q(M_{41})=\mu_5^{N-1}\mu_6\mu_7\mu_{14}\\
&\mathrm{D}_q(M_{14})=\mu_1^{N-1}\mu_2\mu_3\mu_{15} \qquad \mathrm{D}_q(M)=\mu_{16}.
\end{aligned} \tag{5.7}$$

The matrix structure associated with μ_A $(A=1,\ldots,16)$, apart from constant non-zero multiplicative factors, are as follows:

$$\begin{aligned}
&\mu_1=h_N(K^{(1)})\otimes I_N\otimes I_N\otimes I_{N'}\otimes I_{N'}\otimes I_{N'}\\
&\mu_2=g_N(q)\otimes I_N\otimes I_N\otimes I_{N'}\otimes I_{N'}\otimes I_{N'}\\
&\mu_3=I_N\otimes h_N(K^{(2)})\otimes I_N\otimes I_{N'}\otimes I_{N'}\otimes I_{N'}\\
&\mu_4=I_N\otimes g_N(q)\otimes I_N\otimes I_{N'}\otimes I_{N'}\otimes I_{N'}\\
&\mu_5=I_N\otimes I_N\otimes h_N(K^{(3)})\otimes I_{N'}\otimes I_{N'}\otimes I_{N'}\\
&\mu_6=I_N\otimes I_N\otimes g_N(q)\otimes I_{N'}\otimes I_{N'}\otimes I_{N'}\\
&\mu_7=I_N\otimes I_N\otimes I_N\otimes h_{N'}(K^{(4)})\otimes I_{N'}\otimes I_{N'}\\
&\mu_8=I_N\otimes I_N\otimes I_N\otimes g_{N'}^{N'-2+\varepsilon}(q^{1+\varepsilon})\otimes I_{N'}\otimes I_{N'}\\
&\mu_9=I_N\otimes I_N\otimes I_N\otimes I_{N'}\otimes h_{N'}(K^{(5)})\otimes I_{N'}
\end{aligned}$$

$$\mu_{10}=I_N\otimes I_N\otimes I_N\otimes I_{N'}\otimes g_{N'}^{N'-2+\varepsilon}(q^{1+\varepsilon})\otimes I_{N'}$$

$$\mu_{11}=I_N\otimes I_N\otimes I_N\otimes I_{N'}\otimes I_{N'}\otimes h_{N'}(K^{(6)})$$

$$\mu_{12}=I_N\otimes I_N\otimes I_N\otimes I_{N'}\otimes I_{N'}\otimes g_{N'}^{N'-2+\varepsilon}(q^{1+\varepsilon})$$

$$\mu_{13}=\mu_{14}=\mu_{15}=\mu_{16}=I_N\otimes I_N\otimes I_N\otimes I_{N'}\otimes I_{N'}\otimes I_{N'} \tag{5.8}$$

where $K^{(i)}$ for $i=(1,2,3)$ $(i=(4,5,6))$ refer to an $N(N')$ component set of parameters. Notice that the invertibility conditions mentioned following (5.2) and the discussion subsequent to (3.14) and (4.8) show that without loss of generality all the components of $K^{(i)}(i=1,2,\ldots,6)$ may be chosen to be 1.

6. Conclusion

We have developed a systematic procedure for obtaining a finite dimensional representation of the elements of $GL_q(n)$ when q is a primitive Nth root of unity. The method hinges on the invertibility of certain elements of $GL_q(n)$. By making a suitable choice of the variables the commutation relations between elements of $GL_q(n)$ may be reduced to those of $n(n-1)/2$ mutually commuting pairs of Heisenberg–Weyl variables. Now using the representation theory of the Heisenberg–Weyl group in a discrete space of N-vertices of a canonical N-gon on a plane we may obtain a matrix representation of an element of $GL_q(n)$.

To obtain an infinite dimensional representation we express the non-trivial μ-matrices as

$$\mu_{2j-1}=\exp(\mathrm{i}\hat{X}_j) \qquad \mu_{2j}=\exp(\mathrm{i}\hat{P}_j) \tag{6.1}$$

where $j=1,2,\ldots,s$ $(=n(n-1)/2)$. The conjugate pair $(\hat{X}_j,\hat{P}_j)$ satisfy the commutation relation

$$[\hat{X}_j,\hat{P}_j]=\mathrm{i}\theta\nu_{2j-1,2j} \qquad \text{for } q=\exp(-\mathrm{i}\theta) \tag{6.2}$$

and consequently we may choose, for any unimodular q,

$$m_A=\exp\left[\mathrm{i}\sum_{j=1}^{s}(u_{2j-1,A}\hat{X}_j+u_{2j,A}\hat{P}_j)\right]. \tag{6.3}$$

For $GL_q(3)$, Weyers (1990) prescribes the representations essentially in the form (6.3) while requiring the elements u_{ij} to be obtained by solving certain congruence relations. In our language, precisely this was computed by the reduction of the P-matrix to its skew-normal form. Therefore the present technique may be viewed as a systematization of the method of Weyers (1990) for $GL_q(3)$ and its generalization to $GL_q(n)$. It should be noted that we have presented a unified description of the finite and infinite dimensional representations of $GL_q(n)$ in terms of the realizations of (1.1). If we want to use only the infinite dimensional representations of (1.1) then the procedure is much simpler since in that case one can choose $\nu_{2j-1,2j}=1$ for all j $(=1,2,\ldots,s)$ and entries of U are not restricted to integers (see Weyl 1950).

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